A NEW CHART FOR THE SOLUTION OF TRANSMISSION LINE AND POLARIZATION PROBLEMS

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The chart presented in this paper has been described in Ref. 1 where it appeared as an orthographic map of the Poincaré sphere. Its application to transmission lines is believed to be new and since in this respect important properties derive from projective geometry rather than from the relation to the sphere it is proposed to call it the projective chart.

The relations of this chart to Non-Euclidean geometry and to Relativity (Ref. 2 and 3) are interesting and important for understanding its basic properties but since these theories have a reputation of being difficult the projective chart will be considered here as a simple modification of the Smith chart. Fundamental properties will be stated without proof and a selection of possible applications will be given to show the versatility of this new graphical representation.

An important aid in these applications is a transparent overlay with convergent lines and graduations, called the hyperbolic protractor which can be used to measure directly on the chart a special type of distances.

Projective chart

On the Smith chart a reflection coefficient or reflectance w is represented by a point W just as any complex number is represented on the Argand diagram. The distance OW to the origin is the magnitude r of the reflectance and all passive loads are represented by points inside the unit circle Γ. If the line OW cuts Γ at points I and J (Fig. 1) the ratio

$$\frac{WI}{WJ} = \frac{1 + r}{1 - r} \qquad (1)$$

is the VSWR corresponding to the reflectance w.

The modification which leads to the projective chart is to represent the reflectance w by the point $\overline{W}$ with the same phase angle as W but at a distance $\overline{r}$ from the origin given by

$$\overline{r} = \frac{2r}{1 + r^2} \qquad (2)$$

This makes the ratio $\overline{W}I/\overline{W}J$ equal to the square of the VSWR.

If one uses, with the Smith chart, a radial arm carrying a VSWR graduation in decibels the point $\overline{W}$ will be in front of the graduation $2x$ when W is in front of the graduation x. Plotting points on the projective chart or transforming back and forth to the Smith chart is therefore very simple.

The transformation $\mathcal{B}$ from W to $\bar{W}$ can also be obtained by projecting W on a sphere with equator Γ from one of its pole and then by projecting orthogonally from the sphere on the plane of Γ (Ref. 1-2-3) This justifies the construction shown on fig. 2 : $\bar{W}M$ and ON are perpendicular to the radius $O\bar{W}$ and MN goes through W. This can also be used to perform the inverse transformation $\mathcal{B}^{-1}$ from $\bar{W}$ to W.

The circles usually drawn on the Smith chart corresponding to constant resistance or reactance and to constant magnitude or phase of the impedance become on the projective chart straight lines and ellipses as shown on figure 3 . These could be drawn in advance and used as on the Smith chart to plot impedance measurements taken for instance with a bridge.

Distances and angles on the projective chart

Special notions of distance and angle can be introduced on the projective chart which have useful interpretations.

Given two points A,B and the intersections I,J of AB with Γ (fig. 4) the quantity :

$$10 \log_{10} \left(\frac{BI}{BJ} : \frac{AI}{AJ} \right) \qquad (3)$$

will be noted by $\langle AB \rangle$ and called the hyperbolic distance between A and B. It will usually be expressed in decibels as in (3) but can be converted to nepers by substituting $\frac{1}{2} \log_e$ for $10 \log_{10}$.

The quantity (3) deserves the name of "distance" because it satisfies the triangular inequality (which shows that straight lines are geodesics for this particular metric) and because it is additive : if three points A,B,C are on a straight line, in this order

$$\langle AB \rangle + \langle BC \rangle = \langle AC \rangle \qquad (4)$$

The hyperbolic distance between the point $\bar{W}$ and the center of the chart is :

$$\langle O\bar{W} \rangle = 10 \log \frac{1 + \bar{r}}{1 - \bar{r}} = 20 \log \frac{1 + r}{1 - r} \qquad (5)$$

and can be interpreted as the VSWR expressed in decibels.

The expression (3) for the hyperbolic distance does not have to be used in evaluating $\langle AB \rangle$. A scale formed of diverging lines can be traced once for all on a transparent base forming an hyperbolic protractor which is used as follows : the protractor is placed on the chart so that the sides OX, OY of the right angle go through the points I and J (fig. 4) (This is possible in many ways but does not affect the result). The numbers read on the radial lines through A and B respectively are added if A and B are on opposite sides of the radial line marked 0, substracted otherwise. This result divided by 2 is the distance $\langle AB \rangle$. On figure 4 for instance $\langle AB \rangle$ equals (12 + 4)/2 or 8 decibels.

The special type of angle which goes with the hyperbolic metric will be called elliptic. The elliptic angle between the lines $\overline{W}E$ and $\overline{W}F$ is noted by $\langle \overline{W}E, \overline{W}F \rangle$ and can be obtained (Fig. 5) by the following construction : find the point W (if this has not been done) then draw WE and WF produced to their intersections E' and F' with Γ. The elliptic angle is equal to the ordinary angle (OE',OF'). A special elliptic protractor could also be designed to perform this evaluation directly.

Corresponding notions of distance and angle could be introduced directly on the Smith chart: the geodesics are circles orthogonal to Γ, the angle between two of them is represented by the true angle between their tangents at a point of intersection and the "distance" $[AB]$ between two points A and B is most conveniently defined and evaluated by saying that it should be equal to $\langle \overline{A}\overline{B} \rangle$ where $\overline{A}$ and $\overline{B}$ are the images of A and B by the transformation $\mathcal{B}$.

Representation of linear transformers

A transformation which occurs very often because it expresses the effect of a linear transformer on impedance, reflectance or polarization ratio is

$$w' = \frac{aw + b}{cw + d} \qquad (6)$$

where a,b,c,d are complex numbers and w is the quantity which is transformed into w'.

This so called bilinear transformation is represented on the Smith chart by a circular transformation i.e. one which transforms circles into circles and is conformal (preserves angles). It follows that hyperbolic distances are also preserved in the following sense: if A,B are transformed into A',B' while Γ becomes Γ', the distance $[AB]$ defined above is equal to the distance $[A'B']$ measured as if Γ' were the unit circle

$$[AB]_{\Gamma} = [A'B']_{\Gamma'} \qquad (7)$$

the subscript indicating with respect to what circle the "distance" is measured.

The special transformations (6) which preserve the unit circle (lossless transformations of reflectance for instance) are represented on the projective chart by projective transformations. They transform straight lines into straight lines and as a consequence also leave the hyperbolic distances and elliptic angles invariant. The first applications are based on this property.

Change of reference level. Ideal transformer

When impedances are plotted on the Smith chart to convert them into reflectances, they must first be divided by the characteristic impedance of the transmission line to which they will be connected. A change in this characteristic impedance level usually means replotting after a computation (renormalization).

On the projective chart this is unecessary. If the new characteristic impedance is represented by O' instead of the center O (Fig. 6) the new VSWR in decibels is simply the hyperbolic distance $\langle O'\overline{W} \rangle$ while the new phase angle is the elliptic angle between $O'\overline{W}$ and the positive direction $O'P$.

The effect of a change of reference level on reflectances is the same as that of an ideal transformer. One can also visualize the transformer as producing a change of the reflectance $\overline{W}$ into $\overline{W}'$ and the point $\overline{W}'$ can be constructed (Fig. 6) by making

$$(OP, O\overline{W}') = \langle O'P, O'\overline{W} \rangle$$
$$\langle O\overline{W}' \rangle = \langle O'\overline{W} \rangle \quad (8)$$

Reflection coefficient of a stratified medium

The ease in changing reference levels on the projective chart leads to a simple construction for the input reflectance of a lossless stratified medium.

Consider the succession of media $1,2,\ldots,i,\ldots,n$ each with a characteristic impedance Z_i , a thickness d_i and a wave number k_i. From this we deduce a succession of hyperbolic distances

$$u_i = 20 \log_{10}(Z_{i+1}/Z_i) \quad (9)$$

and a succession of angles

$$\alpha_i = k_i\, d_i \quad (10)$$

Then we construct a polygon as follows: we rotate OP clockwise about O through and angle $2\alpha_1$ · obtaining OP_1, we find on OP_1 the point O_1 such that $\langle OO_1 \rangle = u_1$; we rotate O_1P_1 about O_1 clockwise through an elliptic angle $2\alpha_2$, we find on O_1P_2 the point O_2 such that $\langle O_1O_2 \rangle = u_2$ and so on . The end point is O_{n-1} and the last rotation gives $O_{n-1}\, P_n$.

The transformation of reflectance through the stratified medium is represented by the projective transformation which leaves the circle invariant and carries OP onto $O_{n-1}\, P_n$. All characteristics of this transformation: scattering coefficients, image parameters, iterative parameters can be read immediately on this picture. For instance the input reflectance when the last medium is matched is represented by the end point O_{n-1}.

Measurement of reflectance through a lossless junction

The transformation of reflectance from one side of the junction to the other is represented on the projective chart by a transformation

$$T : \quad W \rightarrow W' \quad (11)$$

which is projective, leaves Γ invariant and therefore also the hyperbolic distances and elliptic angles.

Refering to figure 8 we measure on the input side (1) of the junction the reflectances which correspond to four positions of a short circuit taken every eighth of a guide wave length so as to produce at the reference plane (2) the reflectances A,B,C,D shown on figure 8-b. Measurement at the input gives the images A',B',C',D' which lie also on the circle Γ since the junction is lossless but are shown on a different figure (8-c) for clarity.

Since the transformation T is projective the image of O is the intersection O' of A'C' and B'D'. This completes the calibration of the junction.

In order to determine an unknown load W placed in line (2) its image is measured on side (1) of the junction and represented by W' on the projective chart 8-c . Because of the invariance of the hyperbolic distances $\langle O'W' \rangle = \langle OW \rangle$ is the corrected VSWR in decibels and can be evaluated immediately with the protractor while $\langle O'A', O'W' \rangle = (OA, OW)$ is the corresponding corrected phase angle.

Reflectance measurement through a lossy junction

The principle is the same as for the lossless junction : calibration of the junction by measuring the input reflectance for four equispaced positions of a short circuit in the output line, then measurement of the unknown load.

The difference in the correction procedure (Fig. 9) is that it is now simpler to plot the points A',B',C',D' on a Smith chart where they fall on a circle Γ'. The point O' is again the intersection of A'C' and B'D' but instead of using directly the point W' image of the unknown load we convert it to $\overline{W}$ by applying to W' the transformation $\mathcal{B}$ as if Γ' were the unit circle.

The corrected VSWR and phase angle are $\langle O'\overline{W} \rangle$ and $\langle O'A', O'\overline{W} \rangle$ both measured as if Γ' were the unit circle.

These last two applications give the solution to a problem which occurs often in practice. For instance, when experimenting with a new microwave guide, a standard slotted line can be connected to it by means of some junction which, in general, is not matched on either side, is not symmetrical and is often lossy. Improving the junction could be a major problem while correcting for its effect by the usual method would involve finding an equivalent network which is rather complicated when losses occur. The present method is simple and almost as direct for lossy junctions as for lossless ones.

Essential insertion loss of a junction

This concept considered by D. R. Crosby corresponds to the following problem : given a lossy junction, find the minimum insertion loss which could be obtained by placing it between two suitable corrective lossless junctions.

Using the image circle Γ' obtained in the preceding problem we can measure its hyperbolic radius $x = [AB]/2$ (Fig. 10). Then the minimum insertion loss y is a simple function of x which can be tabulated

$$y = 10 \log \frac{1 + 10^{-x/20}}{1 - 10^{-x/20}} \qquad (12)$$

Problems on power

Problems which leads to simple graphical solutions on the projective chart are those where the representation of hermitian forms introduced in reference 3 can be used.

Hermitian forms include such physical quantities as power flow in a transmission line or a polarized plane wave, energy density, voltage or current squared, power absorbed by a load etc. Each is represented, except for a factor, by distances between points on a sphere and planes. By the projection of the sphere on its equatorial plane, which leads to the projective chart, some of these quantities become distances to straight lines.

For instance the ratio of the powers picked up by two probes in a slotted line varies as the ratio of distances from the point $\overline{W}$ representing the reflectance to the tangents D_1 D_2 (Fig. 11) to the unit circle Γ at the points which correspond to the probe positions.

Determination of W or reflectance by three probes measurement is thus reduced to the intersection of two straight lines.

Problems on polarization

Most of them can be deduced from similar problems in transmission lines, the polarization ratio taking the place of the reflectance.

Measurement of polarization by comparing the power picked up by several antennas of known polarization is like the determination of reflectance from a few pickup probes. It reduces to the intersection of a number of planes and in some cases to the intersection of straight lines on the projective chart.

Determination of the polarization of an antenna in the presence of ground or of some linear transformer of polarization, is similar to the measurement of reflectance through a junction and can be solved as above.

Some problems of interference between rays coming from different directions also fall essentially in the same category.

References

1. G. Deschamps, "Geometrical Representation of the Polarization of a Plane Electromagnetic Wave," Proc. IRE, pp. 560-566; May, 1951.
2. G. Deschamps, "Application of Non-Euclidean Geometry to the Analysis of Wave Guide Junctions," URSI-IRE Spring Meeting, 1952.
3. G. Deschamps, "Geometric Viewpoints in the Representation of Wave Guides and Wave Guide Junctions", Proceedings of the Symposium on Modern Network Synthesis, Polytechnic Institute of Brooklyn, New York, 1952.

Acknowledgement

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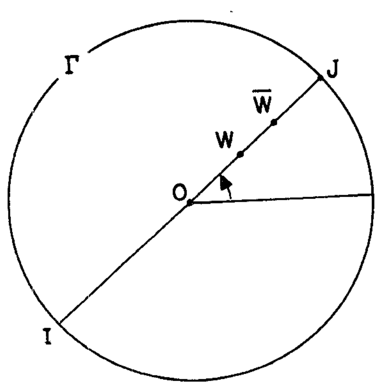


Fig. 1
Relation between the representations of a reflection coefficient (or a polarization ratio) on the Smith Chart (W) and on the projective chart ($\overline{W}$).

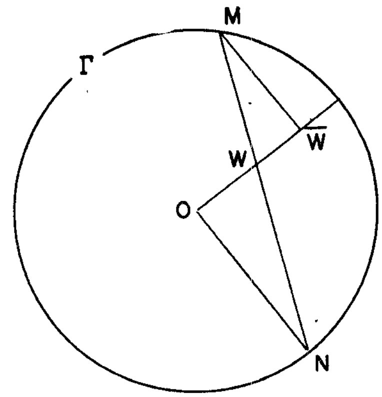


Fig. 2
Transformations $\mathcal{B}$ and $\mathcal{B}^{-1}$: Construction of $\overline{W}$ from W or of W from $\overline{W}$.

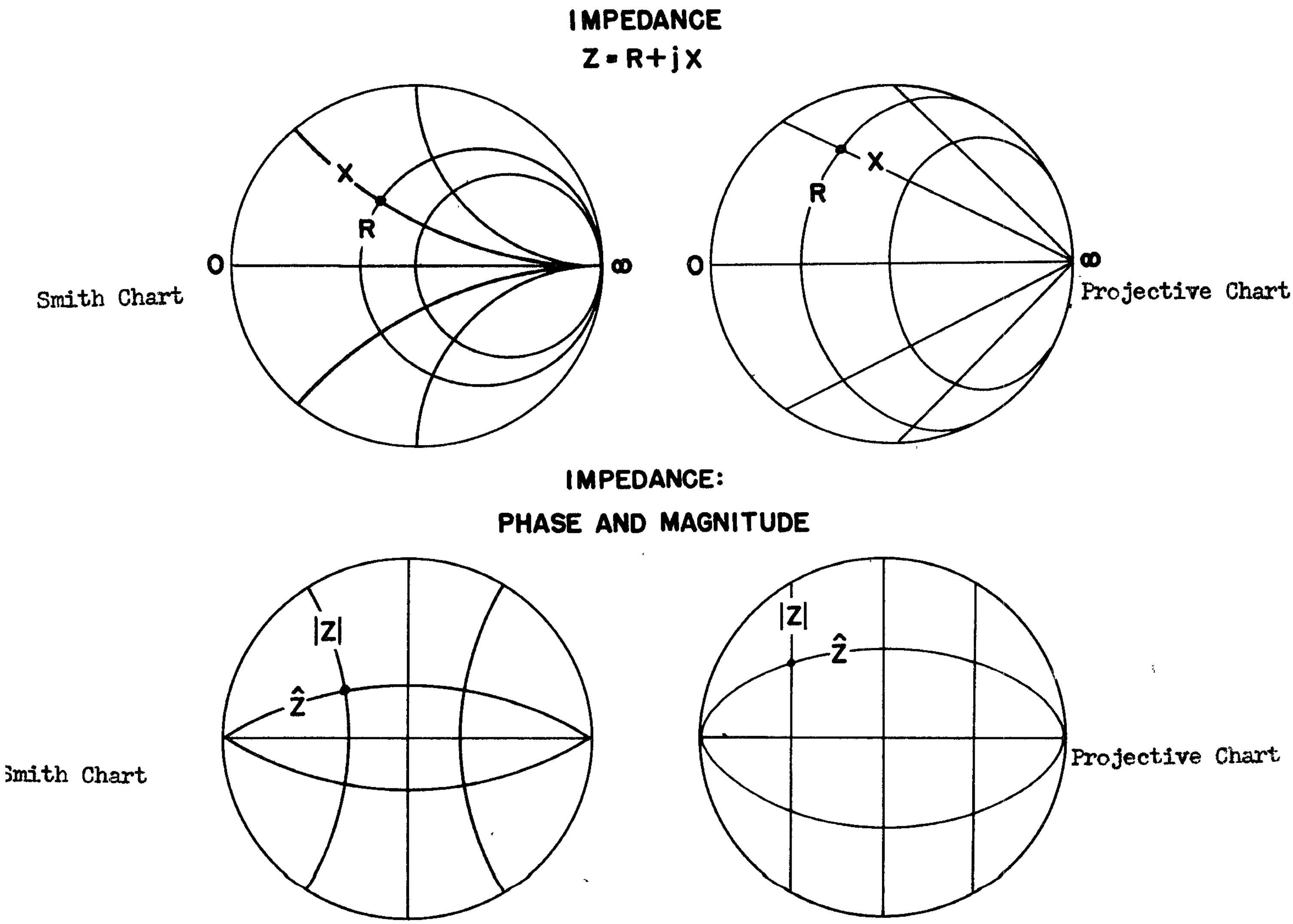


Fig. 3

Loci on the projective chart and on the Smith chart of constant resistance R, reactance X, impedance Magnitude $|Z|$ and impedance phaze $\hat{Z}$.

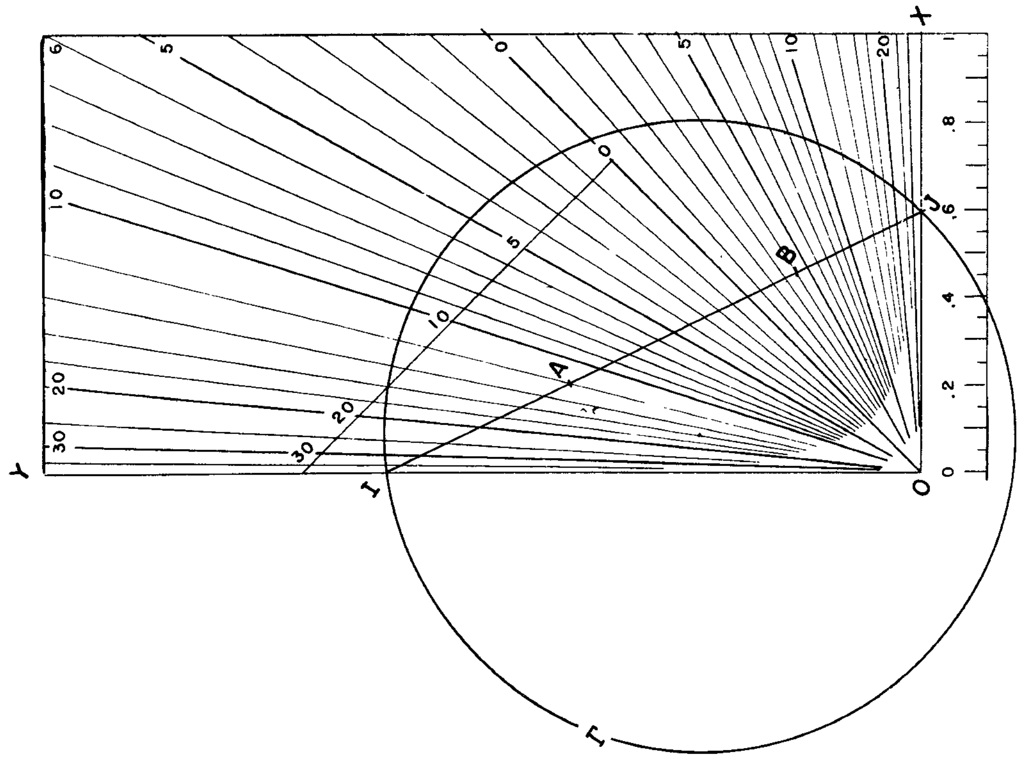
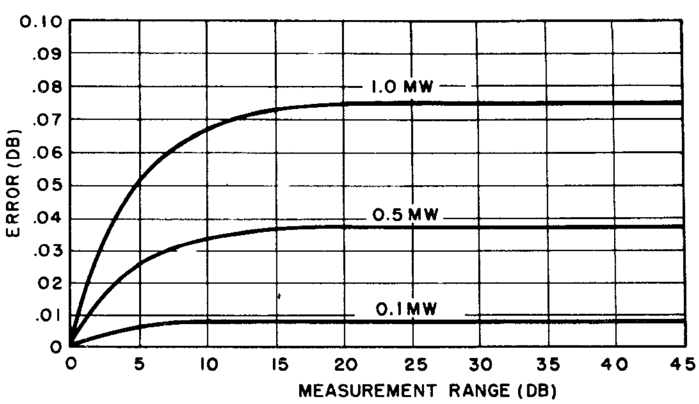


Fig. 4
Definition and evaluation of the hyperbolic distance ⟨AB⟩.



VOLTAGE TRANSFER ERROR IN BARRETTER COUPLING UNIT AS A FUNCTION OF RANGE AND POWER LEVEL

Fig. 5

Evaluation of the elliptic angle $\langle \overline{W}E, \overline{W}F \rangle$.

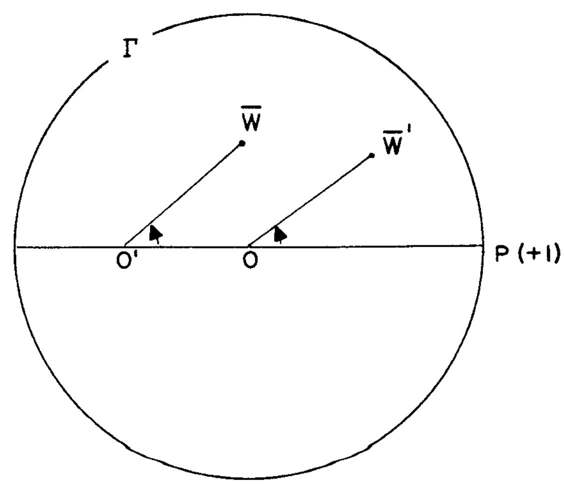


Fig. 6

Change of reference level - Ideal transformer.

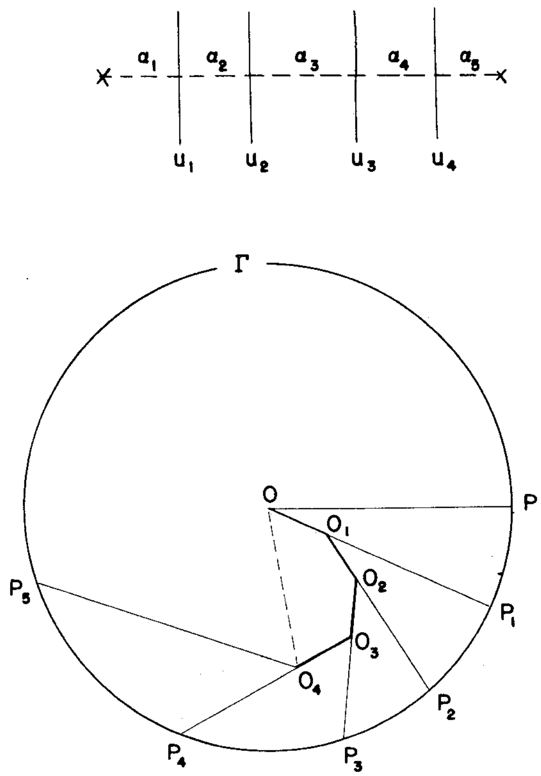
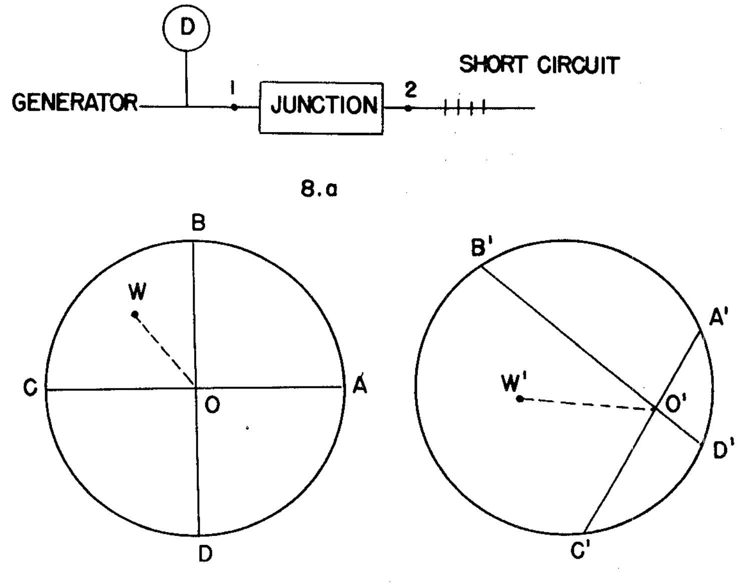


Fig. 7
Transformation of reflectance through a stratified medium.



8.b Fig. 8 8.c

Reflectance measurement through a lossless junction.

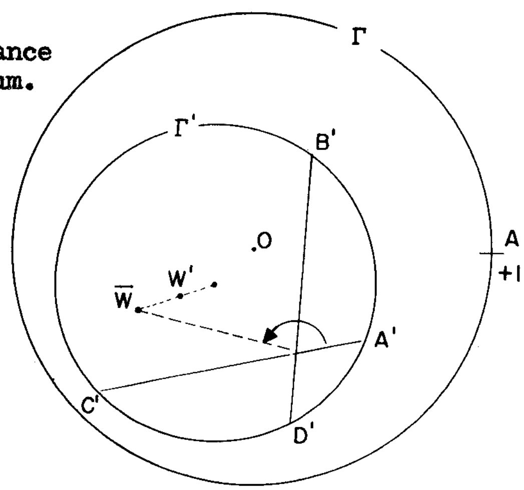


Fig. 9
Reflectance measurement through a lossy junction.

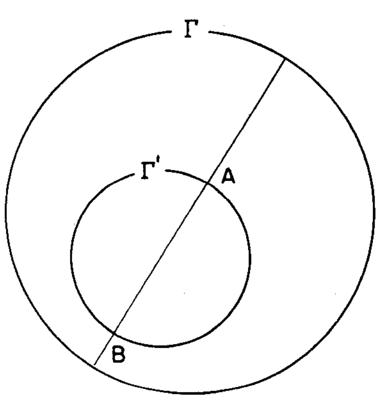


Fig. 10
Evaluation of the essential insertion loss.

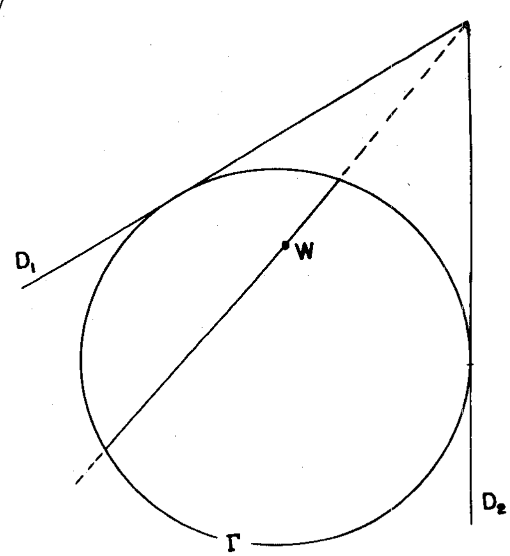


Fig. 11
Locus of W for a given ratio of powers picked up by two probes in a slotted line.